We consider the existence of positive solutions for the Dirichlet problem in two positive parameters $\lambda$ and $\beta$

$$-\Delta_p u = \lambda h(x, u) + \beta f(x, u, \nabla u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

where $h$ and $f$ are continuous nonlinearities satisfying

(H1) $0 \leq \omega_1(x) u^{q-1} \leq h(x, u) \leq \omega_2(x) u^{q-1} \ (1 < q < p);$

(H2) $0 \leq f(x, u, v) \leq \omega_3(x) u^a |v|^b,$

where $\omega_i$, $1 \leq i \leq 3$, are nonnegative weights in the smooth and bounded domain $\Omega \subset \mathbb{R}^N$ and $a, b > 0$. We prove the existence of a region $D$ in the $\lambda\beta$-plane where the Dirichlet problem has at least one positive solution. Our proof rests on a version of the classical result of Tolksdorf and Lieberman, which provides a priori bounds for the gradient.

**Theorem [Tolksdorf-Lieberman]** Assume that $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain and that $g \in L^\infty(\Omega)$. Then there exists a positive constant $K$, depending only on $p$ and $\Omega$, such that

$$\|\nabla u\|_\infty \leq K \|g\|_\infty^{\frac{1}{p-1}}$$

where $u \in C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ is the only weak solution of

$$\begin{cases}
-\Delta_p u = g & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

Let $u_1$ be the first positive eigenfunction of the $p$-Laplacian with weight $\omega_1$, that is,

$$\begin{cases}
-\Delta_p u_1 = \lambda_1 \omega_1 u_1^{p-1} & \text{in } \Omega \\
u_1 = 0 & \text{on } \partial \Omega,
\end{cases}$$

with $\|u_1\|_\infty = 1$ and let $\phi \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ be the solution of the problem

$$\begin{cases}
-\Delta_p \phi = \omega & \text{in } \Omega \\
\phi = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\omega(x) := \max_{i \in \{1, 2, 3\}} \omega_i(x)$.

We denote

$$\gamma := \frac{K \|\omega\|_\infty^{\frac{1}{p-1}}}{\|\phi\|_\infty},$$

where $K$ stands for the constant of the result of Tolksdorf and Lieberman.
Our main result is

**Theorem:** Assume that $h$ and $f$ are continuous and satisfy (H1) and (H2). There exists a region $\mathcal{D}$ in the $\lambda\beta$-plane such that if $(\lambda, \beta) \in \mathcal{D}$ the Dirichlet problem $(P)$ has at least one positive solution $u$ satisfying, for some positive constants $\epsilon$ and $M$:

$$\epsilon u_1 \leq u \leq \frac{M\phi}{\|\phi\|_{\infty}} \quad \text{and} \quad \|\nabla u\|_{\infty} \leq \frac{K\|\omega\|_{\infty}^{\frac{1}{p-1}}M}{\|\phi\|_{\infty}}.$$ 

To prove our result we define for each $u \in C^1$,

$$F^u(x, \xi) := \lambda \omega_1(x)\xi^{q-1} + \lambda \left( h(x, u(x)) - \omega_1(x)u(x)^{q-1} \right) + \beta f(x, u(x), \nabla u(x)) \in C(\Omega).$$

We observe that $F^u(x, u) = \lambda h(x, u) + \beta f(x, u, \nabla u)$.

**Lemma:** There exist positive constants $\epsilon$ and $M$, and a region $\mathcal{D}$ in the $\lambda\beta$-plane such that, if $(\lambda, \beta) \in \mathcal{D}$ and

$$u \in F := \left\{ u \in C^1(\Omega) : 0 \leq u \leq \frac{M\phi}{\|\phi\|_{\infty}} \quad \text{and} \quad \|\nabla u\|_{\infty} \leq \gamma M \right\},$$

then the Dirichlet problem

$$\begin{cases}
-\Delta_p U &= F^u(x, U) \quad \text{in} \quad \Omega \\
U &= 0 \quad \text{on} \quad \partial\Omega.
\end{cases} \quad (P_a)$$

has $u := \epsilon u_1$ and $\overline{u} = \frac{M\phi}{\|\phi\|_{\infty}}$ as an ordered pair of sub- and super-solution and a unique positive solution $U$.

The lemma is proved by applying the sub- and super-solution method, inspired by the seminal paper of Ambrosetti, Brezis and Cerami [1]. The uniqueness of $U$ follows from well-known results proved in [3].

We define the operator

$$T: F \subset C^1(\Omega) \quad u \quad \mapsto \quad C^{1,\alpha}(\overline{\Omega}) \cap W^{1,p}_0(\Omega) \subset C^1(\overline{\Omega}).$$

The existence of a fixed point of $T$ follows from Schauder's fixed point theorem.

**Referências**


